These experiments were conducted at a nominal condition of $M_s = 2.60$, $Re_u = 2.4 \times 10^6/\text{m}$, and $p_1 = 2.933$ kPa with air as the driven gas. In Table 1, driver conditions are listed for the three driver gases. Since driver-created disturbances depend primarily on driver pressure, carbon dioxide produced the most violent disturbances. Measurements of the disturbance environment for correlation to transition data were not obtained.

Results

Transition Reynolds number results from tests at the nominal test condition are presented in Fig. 1 for each driver gas. Data at shock Mach numbers of 2.58-2.63 are included. At each gage location, the data are identified by alternate solid and open symbols and by brackets. Note that the data fall within definite bounds at each gage, indicating no trend in Re_T due to driver gas. Thus, while shock-tube recoil, noise, and vibration disturbances varied according to driver conditions, there was no observed influence of driver-created disturbances on boundary-layer transition. This result is in agreement with Boison's conclusion.[3] For this series of tests, 14% of the data were anomalous, with Re_T values lying both above and below the distinct ranges for each gage in Fig. 1. Nonetheless, these outliers also showed no influence of driver-created disturbances. Another anomaly appears in Fig. 1, namely that Re_T at 6.55 m is larger than at 7.16 m. Such a result is contrary to the transition front concept. For the usual case of approximately constant transition front velocity with a magnitude less than U_s, the expected result would be higher Re_T at 7.16 m. This anomaly was thoroughly investigated, especially regarding the possibility of roughness at the 7.16 m gage, but no definite cause was determined.

Comparison of Re_T in Fig. 1 with data from Ref. 1 indicates that the largest values from the present data[4] are more than double those reported in Ref. 1 for similar wall cooling ratios. This finding prompted further experimentation over a range of M_s. For these tests p_1 was held at 2.933 kPa. Thus Re_u varied inversely with T_w/T_e. The data are presented in Fig. 2 with only the largest Re_T value from the gage locations being shown. Included in the figure are data from Refs. 3, 5, and 6 along with the proposed transition reversal curves for $Re_u = 1.6 \times 10^6/\text{m}$ from Ref. 3.

Observe in Fig. 2 that the present Re_T are significantly larger than the referenced data, except for Ref. 6 at $T_w/T_e < 0.3$. Since the gas-dynamic conditions were similar for the data shown, explanations for the larger Re_T point toward facility differences. Possibly the larger transverse tube dimension for the present study (12.7 cm) was responsible for the higher Re_T compared to Refs. 3 and 5 where the shock tubes were 10.16 cm diam and 3.81×6.85 cm, respectively. This trend of higher Re_T with increasing tube size was given theoretical support by Boehman[2] who showed experimental evidence by comparing data from Refs. 3 and 6 as in Fig. 2 where the Ref. 6 data are seen to be higher. The shock tube in Ref. 6 was 43.18 cm in diameter. These comparisons of results from different size facilities, however, do not provide firm support of the proposed trend. For example, from Fig. 2 consider the magnitude of increase in Re_T ratioed to the increase in transverse dimension. This sensitivity ratio calculated from comparison of the present data with Refs. 3 or 5 is not consistent with the ratio derived from comparison of Refs. 3 and 6. Also, the trend with transverse dimension is reversed when comparing Ref. 6 to the present data for $T_w/T_e > 0.3$, but there could be an Re_u effect in this case. As a result, it cannot be concluded that shock-tube size alone was responsible for the higher Re_T in the present investigation. Probably there were interacting effects of Re_u, M_s, shock tube configuration, and undefined disturbances which contributed to these results.

Another observation from Fig. 2 is that transition reversal occurs at $T_w/T_e \approx 1/3$ for both Ref. 3 and the present data. This result is particularly interesting when consideration is given to the differences in Re_u, Re_T, and shock-tube configurations. Reference 6 data also indicate a change in the Re_T trend at the same level of wall cooling, although the change is not an abrupt reversal. These results tend to support Boehman's[2] calculations wherein certain solutions to the parallel flow stability equations exhibit a "changeover Reynolds number" phenomenon which may be associated with transition reversal. However, further experimental and theoretical work is required to identify and study the parameters controlling this phenomenon in order to aid understanding of reversal for the full range of wall cooling.

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Derivation of the Fundamental Equation of Sound Generated by Moving Aerodynamic Surfaces

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THE fundamental equation of sound generated by arbitrarily moving aerodynamic surfaces was first derived by Ffowcs Williams and Hawkings.[1] These authors based their derivation on the study of mass and momentum balance of a control volume imbedding a mathematical surface(s) exactly corresponding to the real surface(s). They also sketched an alternative method, employing generalized functions, for its derivation. The latter method, later developed by Farassat,[2] is purely mathematical and formal. Yet another, implicit, derivation of the Ffowcs Williams and Hawkings equation was given by Goldstein[3] through the use of free-space Green's function. This Note generalizes Lowson's[4] concept of moving point singularities to moving surface singularities and gives a new derivation of the fundamental equation. The present derivation is based on topological considerations of the underlying space, the fluid medium, and the integral properties of the Dirac delta function. It is simple, direct, physical, and instructive.

For a fluid medium containing arbitrarily moving point mass and force singularities, the equations of continuity and momentum, in their conservation form, may be written as[4]

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$$\frac{\partial \rho}{\partial t}+\frac{\partial}{x_i}(\rho q_i)=\frac{\partial}{\partial t}\sum_{\xi} Q(\xi)\delta(x-\xi) \tag{1}$$

and

$$\frac{\partial}{\partial t}(\rho q_i)+\frac{\partial}{\partial x_j}(\rho q_i q_j+P_{ij})=\sum_{\xi'} F_i(\xi')\delta(x-\xi') \tag{2}$$

where ρ represents the local density of the fluid, t the time, $x_i(i=1,2,3)$ the reference frame fixed with respect to the undisturbed medium, q_i the local fluid velocity, Q the point mass located at the point $x=\xi$, $\delta(x)$ the three-dimensional Dirac delta function with the dimensions of L^{-3} and P_{ij} the compressive stress tensor defined by

$$P_{ij}=p\delta_{ij}-S_{ij} \tag{3}$$

where p is the fluid pressure, δ_{ij} the Kronecker delta, S_{ij} the viscous stress tensor, and F_i the point force located at the point $x=\xi'$. Both ξ and ξ' change with time, Σ denotes summation over all of the point singularities existing in the medium, and the repeated suffix implies the summation over the whole range of the suffix.

In the case of a moving surface, the notion of moving point singularities can be extended to that of a moving surface singularity, achieved as described below.

Let $f(x,t)$ define a continuous functional such that $f(x,t)=0$ represents the equation of the moving surface $\mathbb{S}$ at any time t and that it maps the outside and inside of $\mathbb{S}$ onto $f>0$ and $f<0$, respectively. Further, suppose that the inside of the body has the same state as that of the undisturbed medium. Then, for the moving surface $\mathbb{S}$, the concentrated mass terms in Eq. (1) may be written as

$$\int_{f<0}\delta(x-\xi)\mathrm{d}m$$

where $\mathrm{d}m$ is a differential mass element at the point $x=\xi$, and the integration extends over all the space inside $\mathbb{S}$. Let $H[f(x,t)]$ denote the Heaviside function defined to be unity for $f>0$ and zero for $f<0$, Then the function

$$1-H[f(x,t)]=1 \text{ for } f<0$$
$$=0 \text{ for } f>0$$

This function, which is complementary to the Heaviside function, may be used to extend the region of support, $f<0$, of the above integral to the whole space (WS). The integral on its extented support may be written as

$$\int_{\mathrm{WS}}\rho_0[1-H\{f(\xi,t)\}]\delta(x-\xi)\mathrm{d}V_\xi \tag{4}$$

where ρ_0 is the density of the undisturbed medium and $\mathrm{d}V_\xi$ the volume element at the point $x=\xi$. By the integral properties of the Dirac delta function, one may write the mass source integral [Eq. (4)] as $\rho_0[1-H\{f(x,t)\}]$ or simply as $\rho_0[1-H(f)]$. Substituting the expression

$$\rho_0[1-H(f)] \text{ for } \sum_{\xi} Q(\xi)\delta(x-\xi)$$

in Eq. (1), the continuity equation for a moving surface may, thus, be written as

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho q_i)=\frac{\partial}{\partial t}[\rho_0\{1-H(f)\}] \tag{5}$$

Since $(\partial/\partial t)H(f)=\delta(f)(\partial f/\partial t)$, where $\delta(f)$ is the one-dimensional Dirac delta function, and since $\partial f/\partial t=-v_n|\nabla f|$ (which follows from the fact that $\mathrm{D}f/\mathrm{D}t=0$), where v_n is the local normal velocity of the surface $\mathbb{S}$ and ∇f represents the gradient of f, one may rewrite Eq. (5) as

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho q_i)=\rho_0 v_n|\nabla f|\delta(f) \tag{6}$$

Turning now to the momentum equation (2), $\mathbb{S}$ being a solid surface, there is no momentum flow across $\mathbb{S}$ and the only forces in the fluid medium are, by the hypothesis of the problem, those arising from the tractions at the surface of the body. If p_{ij} is the resultant compressive stress tensor at the surface, defined by

$$p_{ij}=P_{ij}-p_0\delta_{ij} \tag{7}$$

where p_0 is the pressure of the undisturbed field, then

$$F_i=p_{ij}n_j\mathrm{d}S$$

where n_j is the unit vector along the outward normal differential surface element area $\mathrm{d}S$ of the surface $\mathbb{S}$. Accordingly, the point-force terms in Eq. (2) may be written as

$$\int p_{ij}(\xi,t)n_j(\xi,t)\delta(x-\xi)\mathrm{d}S, \qquad \xi\in\mathbb{S} \tag{8}$$

Since the forces are acting on the medium, the surface $\mathbb{S}$ may be taken as the boundary of the fluid as reached from outside the body. As such, since $H(f)=1$ in this region and $f>0$, Eq. (8) may be rewritten as

$$\int p_{ij}(\xi,t)n_j(\xi,t)H[f(\xi,t)]\delta(x-\xi)\mathrm{d}S \tag{9}$$

where $\xi\rightarrow\mathbb{S}$ from the outside.

The functional $f(x,t)$ maps the whole space, the fluid medium plus the body, into the set of real numbers $\mathbb{R}$. Since f is continuous, it can be shown[5] that an exterior neighborhood of $\mathbb{S}$ is mapped into a $f>0$ neighborhood of $f=0$. Since, for $f>0$ near $f=0$, $H(f)$ may be approximated by $\delta(f)\mathrm{d}f$ and since $\delta(f)=0$ for $f\neq 0$, the support of the force integral [Eq. (9)] may be extended from $\mathbb{S}$ to the whole space WS, the integral on its extended support being written as

$$\int_{\mathrm{WS}} p_{ij}(\xi,t)n_j(\xi,t)\delta[f(\xi,t)]\delta(x-\xi)\mathrm{d}S\mathrm{d}f \tag{10}$$

Let the function $f(x,t)=c$, where c is a constant, be chosen as one of the coordinate surfaces of the orthogonal curvilinear system that span the whole space. It can be shown[6] that if $\mathrm{d}n$ is an element of the outward normal to f, then, $\mathrm{d}n$ is given by $\mathrm{d}n=\mathrm{d}f/|\nabla f|$. Since $\mathrm{d}n\mathrm{d}S=\mathrm{d}V$, a volume element, one may write integral (10) as

$$\int_{\mathrm{WS}} p_{ij}(\xi,t)n_j(\xi,t)\delta[f(\xi,t)]\,|\nabla f(\xi,t)|\,\delta(x-\xi)\mathrm{d}V_\xi$$

which, by the integral properties of the Dirac delta function, reduces to the expression $p_{ij}(x,t)n_j(x,t)\delta[f(x,t)]\,|\nabla f(x,t)|$ or, written simply, to $p_{ij}n_j\delta(f)|\nabla f|$. Substituting this expression for the force terms in Eq. (2), the momentum equation for a moving surface is obtained, given as

$$\frac{\partial}{\partial t}(\rho q_i)+\frac{\partial}{\partial x_j}(\rho q_i q_j+P_{ij})=p_{ij}n_j|\nabla f|\delta(f) \tag{11}$$

From now on the steps to be taken to obtain the Ffowcs Williams and Hawkings equation are essentially the same as those used by Lighthill[4] to derive his jet noise theory and shall be repeated here for the sake of completeness.

Differentiating Eq. (6) with regard to t, taking the divergence of Eq. (11), and subtracting the result from the first, one gets

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x_i \partial x_j}(\rho q_i q_i + P_{ij}) + \frac{\partial}{\partial t}[\rho_0 v_n |\nabla f| \delta(f)]$$
$$-\frac{\partial}{\partial x_i}[p_{ij} n_j |\nabla f| \delta(f)]$$

Next, adding and subtracting the term $c_0^2 \nabla^2 \rho$ where c_0 is the velocity of sound in the undisturbed medium and ∇^2 is the Laplacian, one may write the above equation as

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \nabla^2 \rho = \frac{\partial^2}{\partial x_i \partial x_j}(\rho q_i q_j + P_{ij} - c_0^2 \rho \delta_{ij})$$
$$+\frac{\partial}{\partial t}[\rho_0 v_n |\nabla f| \delta(f)] - \frac{\partial}{\partial x_i}[p_{ij} n_j |\nabla f| \delta(f)]$$

Finally, as ρ_0 and p_0 are constants, their time and space derivatives are zero. Therefore, writing ρ as $\tilde{\rho}$ on the left-hand side of the above equation, where $\tilde{\rho}$ is the perturbation density defined by $\tilde{\rho} = \rho - \rho_0$, and writing $P_{ij} - c_0^2 \rho \delta_{ij}$ in the first term on the right as $p_{ij} - c_0^2 \tilde{\rho} \delta_{ij}$, one obtains the Ffowcs Williams and Hawkings equation

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} - c_0^2 \nabla^2 \tilde{\rho} = \frac{\partial^2}{\partial x_i \partial x_j} T_{ij} + \frac{\partial}{\partial t}[\rho_0 v_n |\nabla f| \delta(f)]$$
$$-\frac{\partial}{\partial x_i}[p_{ij} n_j |\nabla f| \delta(f)] \tag{12}$$

where T_{ij} is the Lighthill stress tensor, defined by

$$T_{ij} = \rho q_i q_j + p_{ij} - c_0^2 \tilde{\rho} \delta_{ij} \tag{13}$$

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Transverse Vibrations of Nonuniform Rectangular Orthotropic Plates

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Introduction

INCREASING use is being made of nonisotropic and nonuniform elastic plates in the design of modern missiles, space vehicles, aircraft wings, and numerous composite engineering structures. The investigation presented here gives extensive and accurate results to study the transverse vibrations of a rectangular orthotropic plate of parabolically varying thickness. The governing differential equation of motion is obtained and solved by the Frobenius method to find the first three modes of vibration of a nonuniform rectangular orthotropic plate having different combinations of boundary conditions and for the various values of the taper parameter and length-to-breadth ratio. Some related work is listed in Refs. 1-3.

Equation of Motion

The following differential equation of motion for a nonuniform orthotropic plate is obtained,

$$D_x \frac{\partial^4 w}{\partial x^4} + D_y \frac{\partial^4 w}{\partial y^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + 2\frac{\partial H}{\partial x}\frac{\partial^3 w}{\partial x \partial y^2} + 2\frac{\partial H}{\partial y}\frac{\partial^3 w}{\partial y \partial x^2}$$
$$+2\frac{\partial D_x}{\partial x}\frac{\partial^3 w}{\partial x^3} + 2\frac{\partial D_y}{\partial y}\frac{\partial^3 w}{\partial y^3} + \frac{\partial^2 D_x}{\partial x^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 D_y}{\partial y^2}\frac{\partial^2 w}{\partial y^2}$$
$$+\frac{\partial^2 D_1}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 D_1}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + 4\frac{\partial^2 D_{xy}}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

where

$$D_x = E_1 \frac{h^3}{12}, \quad D_y = E_2 \frac{h^3}{12}, \quad D_1 = E_3 \frac{h^3}{12}, \quad D_{xy} = G_{xy} \frac{h^3}{12}$$

$$H = D_1 + 2D_{xy}$$

Are the rigidity parameters in the appropriate directions of the orthotropy. For convenience we write here,

$$E_1 = \frac{E_x}{(1 - \gamma_{xy}\gamma_{yx})}, \quad E_2 = \frac{E_y}{(1 - \gamma_{yx}\gamma_{xy})}$$

$$E_3 = \gamma_{xy} D_y = \gamma_{yx} D_x$$

Further, w is the transverse deflection of the plate, ρ the mass density per unit volume, h the plate thickness, and E_x, E_y and γ_x, γ_y are, respectively, the Young's modulii and Poisson's

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